Verifying the Kugo–Ojima Confinement Criterion in Landau Gauge Yang–Mills theory

Peter Watson and Reinhard Alkofer

Institute for Theoretical Physics, Tübingen University, Auf der Morgenstelle 14, D-72076 Tübingen, Germany

(February 1, 2008)

Expanding the Landau gauge gluon and ghost two–point functions in a power series we investigate their infrared behavior. The corresponding powers are constrained through the ghost Dyson–Schwinger equation by exploiting multiplicative renormalizability. Without recourse to any specific truncation we demonstrate that the infrared powers of the gluon and ghost propagators are uniquely related to each other. Constraints for these powers are derived, and the resulting infrared enhancement of the ghost propagator signals that the Kugo–Ojima confinement criterion is fulfilled in Landau gauge Yang–Mills theory.

12.38.Aw 12.38.Lg 14.70.Dj 02.30.Rz 11.10.Gh 11.15.Tk

Modifications to the standard framework of quantum field theory are necessary to accommodate the confinement phenomenon in QCD. Here either relaxing the principle of locality or abandoning the positivity of the representation space (or both) is required. As quantum gauge field theories in covariant gauges demand indefinite metric spaces, and as locality is of fundamental importance, it is suggestive to relate confinement to the violation of positivity in Yang–Mills theories: No colored states should be present in the positive definite space of physical states defined by some suitable condition maintaining physical S-matrix unitarity. Exploiting the BRS symmetry of the gauge fixed action a nilpotent BRS-charge $\mathcal{Q}_\mathcal{B}$ in an indefinite metric space $\mathcal{V}$ can be constructed. The subspace $\mathcal{V}_\mathrm{p} = \mathrm{Ker}\,\mathcal{Q}_\mathcal{B}$ is defined by those states which are annihilated by the BRS charge $\mathcal{Q}_\mathcal{B}$. Positivity is then proved for physical states [1] (see also refs. [2,3]) which are given by the cohomology $\mathcal{H}(\mathcal{Q}_\mathcal{B}, \mathcal{V}) = \mathrm{Ker}\,\mathcal{Q}_\mathcal{B}/\mathrm{Im}\mathcal{Q}_\mathcal{B}$, the covariant space of equivalence classes of BRS-closed modulo BRS-exact states.

Longitudinal and timelike gluons form BRS quartets together with ghosts and are thus unphysical. At the same time the global symmetry $J^a_{\mu,\nu}$ corresponding to gauge transformations generated by $\theta^a(x) = a^a_\mu x^\mu$ is spontaneously broken quite analogous to the displacement symmetry in QED. The identification of BRS-singlet states with color singlets is non-perturbatively possible if this global symmetry is restored [1,2,4]. The condition for a dynamical restoration of this symmetry is the Kugo–Ojima confinement criterion which is a requirement on the infrared behavior of a four-point Green's function. In Landau gauge, a sufficient condition for this criterion is that the nonperturbative ghost propagator is more singular than a massless pole in the infrared [4]:

$$D^{ab}_G(p) = \delta^{ab}\frac{G(p^2)}{p^2}\,, \quad \text{with} \quad G(p^2) \overset{p^2\to 0}{\longrightarrow} \infty\,. \tag{1}$$

This mechanism is correlated to the derivation [5] of the Oehme–Zimmermann superconvergence relations [6] from Ward–Takahashi identities. These superconvergence relations formalize a long known contradiction between asymptotic freedom and the positivity of the spectral density for transverse gluons in the covariant gauge.

The violation of positivity of transverse gluons is unambiguously established by a variety of independent non-perturbative studies of the gluon propagator [7]. Furthermore, recent lattice calculations investigate the Kugo–Ojima criterion directly [8]. As infrared singularities are anticipated, also non-perturbative *continuum* methods, besides lattice calculations, are necessary to investigate this and related pictures of confinement in more detail.

The Green's functions of a quantum field theory obey the Dyson–Schwinger equations. For realistic theories this infinite tower of integral equations cannot be solved directly, and truncations are necessary to obtain (numerical) solutions. The ones for QCD propagators can then be used to formulate a successful hadron physics phenomenology, see *e.g.* the recent reviews [3,9].

The results of two recent truncation schemes [10,11] suggest that the Kugo–Ojima confinement criterion for the ghost propagator and the violation of positivity for transverse gluon correlations are fulfilled. Obviously, the question arises whether these results are sensitive to the employed truncation schemes.

In this letter we will exploit the general structure of the simplest QCD Dyson–Schwinger equation, the one for the ghost two–point function, to relate the infrared behavior of gluon and ghost propagators. Thereby *the Kugo–Ojima confinement criterion will be verified by purely analytical methods.* Using the rather unique properties of the ghost-gluon vertex in the Landau gauge and assuming the applicability of a powerlaw ansatz in the infrared the ghost Dyson–Schwinger equation admits a completely general qualitative analysis. Instead of truncating the system, the general forms for the unknown functions constrained by various consistency arguments, notably multiplicative renormalizability, are input into the equation under an asymptotic expansion in the infrared momentum scale, *c.f.* the method described in ref. [12].

We start by providing some basic notations. The general form of the gluon propagator in Landau gauge is purely transverse and is given by [3,9]

$$D^{ab}_{\mu\nu}(p) = \delta^{ab}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Z(p^2)}{p^2}\,, \tag{2}$$

which defines the gluon renormalization function $Z(p^2)$. The ghost renormalization function $G(p^2)$ is established in eq. (1). We define the ghost-gluon vertex

$$\Gamma^{abc}_\mu(p_1,p_2,p_3) = -igf^{abc}\Gamma_\mu(p_1,p_2,p_3) \tag{3}$$

with antighost (p_1), ghost (p_2) and gluon (p_3) momentum incoming, *i.e.* $p_1+p_2+p_3=0$. At tree-level the dressing functions reduce to

$$Z(p^2) = G(p^2) = 1, \quad \Gamma_\mu(p_1,p_2,p_3) = p_{1\mu}\,. \tag{4}$$

These renormalized dressing functions are related to the bare ones by the usual renormalization constants Z_3 (gluon wave function), $\widetilde{Z}_3$ (ghost wave function) and $\widetilde{Z}_1$ (ghost-gluon vertex). In Landau gauge one has $\widetilde{Z}_1 = 1$ [13,14], *i.e.*, the ghost-gluon vertex has not to be renormalized .

The renormalized ghost dressing function $G(p^2)$ is a dimensionless function of a single variable. In pure Yang–Mills theory there are only two scales present: the external scale p^2 and the renormalization scale μ^2. Therefore G is a function of the ratio p^2/μ^2 only. As the external scale vanishes this function can be written as an asymptotic expansion. Renormalizability implies that it must also depend on the renormalized coupling g. This motivates the power series ansatz

$$G(p^2;\mu^2) = \sum_n d_n\left(\frac{p^2}{\mu^2}\right)^{\delta_n}\,, \tag{5}$$

where the coefficients d_n and the exponents δ_n are, at least in principle, functions of g. We will see below that the exponents δ_n will be independent of g. Furthermore, as will be shown later on, the series (5) has a lowest power δ_0 such that as the external scale p^2 vanishes the ghost dressing function $G(p^2)$ is well approximated by a powerlaw. Multiplicative renormalizability allows to derive the renormalization group (RG) equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma_{\rm G}\right)G(p^2;\mu^2) = 0 \tag{6}$$

with $\beta(g)$ being the beta function, and

$$\gamma_{\rm G} = \mu\frac{\partial\log\widetilde{Z}_3^{-\frac{1}{2}}}{\partial\mu} \tag{7}$$

is related to the ghost anomalous dimension. Inserting the series (5) into eq. (6), and exploiting that p^2 is a free (though small) parameter one obtains

$$-d_n\delta_n + \beta(g)\left(\frac{\partial d_n}{\partial g} + d_n\log(\frac{p^2}{\mu^2})\frac{\partial\delta_n}{\partial g}\right) + d_n\gamma_{\rm G} = 0. \tag{8}$$

Since the logarithm has no matching factor the exponents δ_n must be independent of the renormalized coupling ($\partial\delta_n/\partial g = 0$). Using $\beta(g) = -g(2\gamma_{\rm G}+\gamma_{\rm A})$ with $\gamma_{\rm A} = \mu\partial\log Z_3^{-\frac{1}{2}}/\partial\mu$ the resulting differential equation for the coefficients d_n has the general solution

$$d_n = \text{const.}\; g^{-\frac{2(\delta_n+\gamma_{\rm G})}{2\gamma_{\rm G}+\gamma_{\rm A}}}\,. \tag{9}$$

Expanding the renormalized gluon dressing function

$$Z(p^2;\mu^2) = \sum_n e_n\left(\frac{p^2}{\mu^2}\right)^{\epsilon_n} \tag{10}$$

one arrives in a completely analogous way at

$$e_n = \text{const.}\; g^{-\frac{2(\epsilon_n+\gamma_{\rm G})}{2\gamma_{\rm G}+\gamma_{\rm A}}}\,. \tag{11}$$

In the renormalization process we have to specify the corresponding normalization conditions. We require that the unrenormalized functions reduce to a constant at asymptotically large momenta. Exploiting additionally multiplicative renormalizability and using eq. (5) one obtains the formal relation

$$\widetilde{Z}_3(\mu^2,\Lambda^2) \propto \left(\sum_n d_n\left(\frac{\Lambda^2}{\mu^2}\right)^{\delta_n}\right)^{-1}\,. \tag{12}$$

This equation is only well-defined if the corresponding series makes sense, *e.g.* if it is resumable by some method. This is the main assumption throughout the presented derivation. One consistency check is provided by inserting the general solution for the coefficients d_n (9) and rewriting the renormalized coupling g in terms of the scale μ. This yields

$$\widetilde{Z}_3(\mu^2,\Lambda^2) \propto \left(\frac{\Lambda^2}{\mu^2}\right)^{\gamma_{\rm G}} \tag{13}$$

which is precisely the correct dependence to agree with the definition of $\gamma_{\rm G}$. A completely analogous argument applies to the gluon renormalization constant Z_3.

The tensor structure of the ghost-gluon vertex reads

$$\Gamma_\mu(p_1,p_2,p_3) = p_{1\mu}A(p_i^2) + p_{3\mu}B(p_i^2)\,. \tag{14}$$

In Landau gauge the ghost-gluon vertex becomes bare when the ghost momentum p_2 vanishes [13,14]:

$$\Gamma_\mu(p_1,p_2,p_3) = p_{1\mu}\,(A-B) - p_{2\mu}B \stackrel{p_2\to 0}{\longrightarrow} p_{1\mu}\,. \tag{15}$$

Thus, as $p_2 \to 0$ the functions A and B cannot be singular. This is in accordance with the fact that the renormalization constant $\widetilde{Z}_1 = 1$.

The RG equation for the vertex function can be split into two parts, one for each of the tensor components. It is only necessary to consider the equation for A, and since $\widetilde{Z}_1 = 1$, it is identical to the RG equation for the

coupling. Writing the vertex function as a series has the complication of several momentum scales. For our considerations it is sufficient to simply extract the scale and coupling dependence

$$A(p_i^2;\mu^2) = \sum_k a_k(p_1^2,p_2^2)\lambda_k \left(\frac{p_3^2}{\mu^2}\right)^{\kappa_k}, \tag{16}$$

with

$$\lambda_k = \left(g^2\right)^{-\frac{\kappa_k}{\gamma_\mathrm{A}+2\gamma_\mathrm{G}}} . \tag{17}$$

We must renormalize the ghost-gluon vertex at the point $p_2 = 0$ since this corresponds to a zero finite renormalization from $\widetilde{Z}_1 = 1$. Knowing the $p_2 \to 0$ limit of the full vertex does not constrain A. (When $p_2 = 0$ one only has $A - B = 1$.) With this given limit and the additional requirement that the ghost equation should provide a consistent solution one infers that

$$A(p_1^2 = p_3^2 = \Lambda^2, p_2^2 = 0; \mu^2) = \sum_k a_k\lambda_k \left(\frac{\Lambda^2}{\mu^2}\right)^{\kappa_k} \tag{18}$$

is a finite positive constant. Asymptotic freedom requires that $\gamma_\mathrm{A} + 2\gamma_\mathrm{G} > 0$. In the unrenormalized theory the vertex should be bare when the unrenormalized coupling vanishes because $\widetilde{Z}_1 = 1$. Therefore $\kappa_k \le 0$ with the highest value $\kappa_0 = 0$.

The ultraviolet region of the ghost Dyson–Schwinger equation corresponds to a small (albeit non-vanishing) ghost momentum p_2 and all other momenta large. For the following discussions it is helpful to write

$$A(p_i^2;\mu^2)_{p_2\to 0} = \sum_{k,m} b_k(p_1^2)\lambda_k \left(\frac{p_3^2}{\mu^2}\right)^{\kappa_k} \left(\frac{p_2^2}{p_3^2}\right)^{\rho_m} \tag{19}$$

where the $\rho_m \ge 0$ and $\rho_0 = 0$.

The full Euclidean ghost Dyson–Schwinger equation can be written after having performed the two trivial angular integrals as

$$\begin{aligned} G^{-1}(x) &= \widetilde{Z}_3 \\ &-\frac{C_A g^2}{16\pi^2}\int_0^{\Lambda^2} dy Z(y)\frac{2}{\pi}\int_0^\pi d\theta \sin^4\theta \frac{G(z)}{z} A(z,x,y) , \end{aligned} \tag{20}$$

where θ is the angle between the two internal loop momenta corresponding to $y = q^2$ and $z = (p-q)^2$ with $x = p^2$. $C_A = N_c$ is the resulting color factor. When using the expansions (5) and (10) the angular integration splits the radial integral into two parts, the lower part I_{gh} $(0 \to x)$ and the upper part J_{gh} $(x \to \Lambda^2)$. The expression derived from the upper limit will be some combination of factors with the ratio x/Λ^2 to some power. The effect of an exact angular integration is to generate a hypergeometric function whose argument is x/y and which can be expanded in a convergent series with positive integer powers of x/y starting with unity. Thus, the z-dependence of the vertex function is unimportant.

The most important aspect is that the ghost-gluon vertex reduces to its tree-level form as the in-ghost momentum vanishes. In the ultraviolet part of the integral this is precisely the case: the in-ghost momentum is the infrared external scale x. Therefore the vertex function can be written in the form of eq. (19).

Using the series for G and Z, it is possible to write the ghost equation with the most general vertex function as

$$\begin{aligned} \left(\sum_n d_n \left(\frac{x}{\mu^2}\right)^{\delta_n}\right)^{-1} &= \widetilde{Z}_3 \\ -\frac{C_A g^2}{16\pi^2}\sum_{j,k} e_j d_k \left(\frac{x}{\mu^2}\right)^{\epsilon_j+\delta_k} &\left(I_{gh}(\epsilon_j,\delta_k) + J_{gh}(\epsilon_j,\delta_k)\right) . \end{aligned} \tag{21}$$

where I_{gh} and J_{gh} represent the lower and upper regions of the integral, respectively. The integrals I_{gh} must contain factors characterized by the κ_k but there are no restrictions on its form except that the vertex must have positive powers of the coupling. As the integral I_{gh} is very difficult to compute we will not be able to proceed by studying its general properties.

It is frequently asserted that only the infrared integrals of Dyson–Schwinger equations can contribute to the renormalized propagator functions on the left hand side because this is the only fully renormalizable possibility. Indeed, one can show that upper integral J_{gh} will not contribute directly to the propagator functions [15]. It is the renormalization constant $\widetilde{Z}_3$ that provides the key to the infrared behavior of the propagators, and only the ultraviolet integral can yield the ultraviolet divergent part of the ghost equation. First we note that $J_{gh}(\epsilon_j,\delta_k)$ can be expanded as

$$J_{gh} = \sum_{l,m,q} J_{j,k,l,m,q}\left(\frac{\Lambda^2}{x}\right)^{\epsilon_j+\delta_k-\rho_m-q} \lambda_l \left(\frac{\Lambda^2}{\mu^2}\right)^{\kappa_l} , \tag{22}$$

where $\rho_m \ge 0$ represents the kinematical content of the vertex function, and $q = 0, 1, \ldots$ denotes the order of the angular integration series. As $\widetilde{Z}_3$ has to be independent of x one deduces from eqs. (21,22) that only $\rho_0 = q = 0$ will contribute. All other terms involve x/Λ^2 to a positive power and subsequently vanish:

$$\begin{aligned} &\widetilde{Z}_3(\mu^2,\Lambda^2) = \\ &\quad \frac{C_A g^2}{16\pi^2} \sum_{j,k,l} e_j d_k J_{j,k,l,0,0}\lambda_l \left(\frac{\Lambda^2}{\mu^2}\right)^{\epsilon_j+\delta_k+\kappa_l} . \end{aligned} \tag{23}$$

Using that expression (18) is a constant, employing the expansion (12) and multiplying through with the left hand side yields

$$\frac{C_A g^2}{16\pi^2}\sum_{i,j,k} e_j d_i d_k J_{0,0,0}(\epsilon_j,\delta_k)\left(\frac{\Lambda^2}{\mu^2}\right)^{\epsilon_j+\delta_i+\delta_k} = \text{const.} \tag{24}$$

Since ϵ_0 and δ_0 are the unique lowest powers, and the term with $i = j = k = 0$ has nothing to compete with it must be responsible for the non-vanishing part of the left hand side of this equation. Therefore it must be independent of Λ^2/μ^2, and we obtain

$$\epsilon_0 = -2\delta_0 \, , \tag{25}$$

i.e. we have shown that *the leading infrared powers of the gluon and the ghost propagator are uniquely related.* Note that this result is valid even if corresponding series are not convergent. We have simply matched the coefficients of an asymptotic series. Therefore, our underlying assumption is that the employed series have to exist in the sense of being, at least, asymptotic ones.

Eq. (24) constrains the values of the related leading powers. To do this, one needs to explicitly evaluate the integrals $J_{j,k,0,0,0}$ which are readily calculated from the full ghost equation with a bare vertex stripping off the terms that depend on the external scale x. Concentrating on the leading powers we obtain:

$$-\frac{3}{4}\frac{C_A g^2}{16\pi^2} e_0 d_0^2 \frac{1}{\delta_0} > 0. \tag{26}$$

To yield a positive ghost renormalization function (*i.e.* to keep the negative-norm property of the ghost states) the corresponding leading order coefficients e_0 and d_0 must be positive. As the coupling is positive this yields:

$$\delta_0 < 0 \, . \tag{27}$$

This relation implies that *the Kugo–Ojima confinement criterion*, see eq. (1), *is fulfilled.*

A lower bound on δ_0 can be derived from the lower bound of the integral in the ghost equation. As this bound is considerably weaker as the one following from a generalized masslessness condition (see eq. (192) in ref. [3]) $\delta_0 > -1$, we will not present its derivation here. This implies $2 > \epsilon_0 > 0$: The gluon propagator is suppressed in the infrared as compared to its tree-level form.

In summary, we have investigated the general structure of the ghost Dyson–Schwinger equation in Landau gauge. A general series form for the propagator and vertex functions has been employed. No specialized ansatz for the ghost-gluon vertex has been used. Care has been taken not to use approximations that destroy the generality of the results. Multiplicative renormalizability, and especially the non-renormalization of the ghost-gluon vertex, have placed the constraints to relate uniquely the leading infrared behavior of the gluon and the ghost propagator.

We have found an infrared enhanced ghost propagator thereby verifying the Kugo–Ojima confinement criterion [1,4] directly. Due to the relation (25) this corresponds to an infrared suppressed gluon propagator. For a non-integer leading infrared power this implies that positivity is violated and the Oehme–Zimmermann superconvergence relation [6] is fulfilled non-perturbatively. Considering only the ghost equation as in this letter an infrared constant gluon propagator cannot be excluded. To investigate this question one has to consider additionally the gluon Dyson–Schwinger equation [15].

Acknowledgements

We want to express our gratitude to M.R. Pennington who has accompanied the whole research presented here with helpful discussions. We thank C. Fischer, S. M. Schmidt and L. von Smekal for their valuable remarks.

P. W. acknowledges the University of Durham and the CSSM, University of Adelaide, where major parts of this research have been done.

This research has been supported by DFG under contract Al 279/3-3.

[1] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66** (1979) 1.

[2] N. Nakanishi and I. Ojima, "Covariant Operator Formalism Of Gauge Theories And Quantum Gravity," *World Scientific lecture notes in physics 27 (1990).*

[3] R. Alkofer and L. von Smekal, Physics Reports, in press [hep-ph/0007355].

[4] T. Kugo, Int. Symp. on BRS symmetry, Kyoto, Sept. 18–22, 1995, hep-th/9511033.

[5] K. Nishijima, Czech. J. Phys. **46** (1996) 1; M. Chaichian and K. Nishijima,hep-th/9909158; and references therein.

[6] R. Oehme and W. Zimmermann, Phys. Rev. **D 21** (1980) 471; 1661.

[7] J. E. Mandula, Phys. Rept. **315** (1999) 273.

[8] S. Furui and H. Nakajima, hep-lat/0012017; and references therein.

[9] C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. **45** (2000) S1.

[10] L. von Smekal, R. Alkofer and A. Hauck, Phys. Rev. Lett. **79** (1997) 3591; Annals Phys. **267** (1998) 1.

[11] D. Atkinson and J. C. Bloch, Phys. Rev. **D 58** (1998) 094036; Mod. Phys. Lett. **A13** (1998) 1055.

[12] H. Pagels, Phys. Rev. **D 15** (1977) 2991.

[13] J. C. Taylor, Nucl. Phys. **B33** (1971) 436.

[14] W. Marciano and H. Pagels, Phys. Rept. **36** (1978) 137.

[15] P. Watson and R. Alkofer, in preparation.